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THE RING OF DIFFERENTIAL OPERATORS ON AN AFFINE TORIC VARIETY

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1. Introduction and Motivation

The ring of differential operators was introduced by Grothendieck [6]. Although it may be ugly in general [1], the ring of differential operators on an affine toric variety has some good features. The aim of this article is to exhibit some of them, in particular, a good structure of the spectrum of its graded ring (with respect to the order filtration) on a scored affine toric variety. In the final section, we consider the characteristic varieties of critical modules, which live in the spectrum of the graded ring.

Let $A := (\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = (a_{ij})$ be a $d \times n$ matrix with coefficients in $\mathbb{Z}$. We sometimes identify A with the set of its column vectors. We assume that $\mathbb{Z}A = \mathbb{Z}^d$, where $\mathbb{Z}A$ is the abelian group generated by A.

For $\boldsymbol{\beta} \in \mathbb{C}^d$, the **$A$-hypergeometric system** with parameter $\boldsymbol{\beta}$ is defined by

$$M_A(\boldsymbol{\beta}) := D/DI_A(\partial_x) + D\langle A\theta - \boldsymbol{\beta}\rangle,$$

where

- $D := \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}\rangle$: the nth Weyl algebra.
- $I_A(\partial_x) := \langle \partial_x^{\boldsymbol{u}} - \partial_x^{\boldsymbol{v}} \,:\, A\boldsymbol{u} = A\boldsymbol{v},\, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{N}^n\rangle$: the toric ideal.
- $\langle A\theta - \boldsymbol{\beta}\rangle := \sum_{i=1}^d \mathbb{C}[\theta] \sum_{j=1}^n (a_{ij}\theta_j - \beta_i)$: the Euler operators
- $\mathbb{C}[\theta] := \mathbb{C}[\theta_1, \dots, \theta_n], \qquad \theta_j = x_j \partial_{x_j}$.

After the systematic study of the A-hypergeometric systems by Gel'fand and his collaborators ([3], [4], etc.), the systems are also known as GKZ-hypergeometric systems.

In this section, we see that the ring of differential operators on an affine toric variety naturally arises as the algebra of contiguity operators of A-hypergeometric systems [19].

Suppose that $P \in D$ satisfies

- $I_A(\partial_x)P \subseteq DI_A(\partial_x)$,
- $\langle A\theta - \boldsymbol{\beta} - \boldsymbol{a}\rangle P = P\langle A\theta - \boldsymbol{\beta}\rangle$.

Then there exists a D-module homomorphism

$$M_A(\boldsymbol{\beta} + \boldsymbol{a}) \xrightarrow{\times P} M_A(\boldsymbol{\beta})$$

or P is a **contiguity operator** shifting parameters by $\boldsymbol{a}$

$$\mathrm{Hom}_D(M_A(\boldsymbol{\beta}), \mathcal{O}) \ni f \mapsto Pf \in \mathrm{Hom}_D(M_A(\boldsymbol{\beta} + \boldsymbol{a}), \mathcal{O}),$$

where $\mathcal{O}$ is a D-module of some functions; $\mathrm{Hom}_D(M_A(\boldsymbol{\beta}), \mathcal{O})$ may be identified with the space of solutions of $M_A(\boldsymbol{\beta})$ in $\mathcal{O}$.

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Consider the algebra of contiguity operators

$$\{P \in D \,:\, I_A(\partial_x)P \subseteq DI_A(\partial_x)\}.$$

Since $I_A(\partial_x)$ operates trivially on $M_A(\boldsymbol{\beta})$ for all $\boldsymbol{\beta}$, we consider

$$\mathrm{Sym}_A := \{P \in D \,:\, I_A(\partial_x)P \subseteq DI_A(\partial_x)\}/DI_A(\partial_x).$$

Then Sym_A is an algebra, and

$$\mathrm{Sym}_A = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} \mathrm{Sym}_{A,\boldsymbol{a}},$$

where

$$\mathrm{Sym}_{A,\boldsymbol{a}} = \{P \in \mathrm{Sym}_A \,:\, \langle A\theta\rangle P = P\langle A\theta + \boldsymbol{a}\rangle\}.$$

Let ι be the anti-automorphism of D defined by

- $\iota(x_j) = \partial_{x_j}, \quad \iota(\partial_{x_j}) = x_j \qquad (\forall j)$,
- $\iota(PQ) = \iota(Q)\iota(P)$.

Note that $\iota(\theta_j) = \iota(x_j\partial_{x_j}) = \iota(\partial_{x_j})\iota(x_j) = x_j\partial_{x_j} = \theta_j$.

Then

$$\begin{aligned} \iota(\mathrm{Sym}_A) &= \iota(\{P \in D \,:\, I_A(\partial_x)P \subseteq DI_A(\partial_x)\})/\iota(DI_A(\partial_x)) \\ &= \{P \in D \,:\, PI_A(x) \subseteq I_A(x)D\}/I_A(x)D. \end{aligned}$$

This is identified with the ring $D(R_A)$ of differential operators on the affine toric variety defined by A (cf. [10, Theorem 5.13]).

2. Definitions

In this section, we give some basic definitions. Let $\mathbb{N}A$ be the monoid generated by A. Let R_A denote the semigroup algebra of $\mathbb{N}A$, i.e.,

$$R_A := \mathbb{C}[\mathbb{N}A] = \bigoplus_{\boldsymbol{a}\in\mathbb{N}A} \mathbb{C}t^{\boldsymbol{a}} \subseteq \mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}].$$

Here and hereafter we use multi-index notation; $t^{\boldsymbol{a}} = t_1^{a_1}t_2^{a_2}\cdots t_d^{a_d}$ for $\boldsymbol{a} = {}^t(a_1, a_2, \ldots, a_d)$. The ring of differential operators of the Laurent polynomial ring $\mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]$ equals

$$\mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]\langle\partial_1, \ldots, \partial_d\rangle, \quad \text{where } \partial_i = \partial_{t_i}.$$

Then the ring of differential operators of R_A (or on the affine toric variety defined by A) can be given as a subalgebra of the ring of differential operators on the big torus:

$$D(R_A) = \{P \in \mathbb{C}[t_1^{\pm1}, \ldots, t_d^{\pm1}]\langle\partial_1, \ldots, \partial_d\rangle \,:\, P(R_A) \subset R_A\}.$$

Let $D_k(R_A)$ be the subspace of differential operators of order less or equal to k in $D(R_A)$. Then the graded ring with respect to the order filtration $\{D_k(R_A)\}$ is commutative:

$$G := \mathrm{Gr}\, D(R_A) = \bigoplus_{k=0}^{\infty} D_k(R_A)/D_{k-1}(R_A) \subseteq \mathbb{C}[t_1^{\pm}, \ldots, t_d^{\pm}, \xi_1, \ldots, \xi_d],$$

where ξ_i denotes the image of ∂_i.

3. Finiteness

In general, the ring of differential operators on an affine variety may be neither left or right Noetherian nor finitely generated as an algebra [1]. In this section, we give some results on finiteness of $D(R_A)$.

Theorem 3.1 ([20]). *$D(R_A)$ is a finitely generated $\mathbb{C}$-algebra.*

Theorem 3.2 ([18]). (1) *$D(R_A)$ is right Noetherian.*
(2) *$D(R_A)$ is left Noetherian if $\mathbb{N}A$ is* S_2.

In [18], we also gave a necessary condition for $D(R_A)$ being left Noetherian.

Definition 3.3. A semigroup $\mathbb{N}A$ is S_2 if $\mathbb{N}A = \bigcap_{\sigma\colon \text{facet of } \mathbb{R}_{\geq 0}A} [\mathbb{N}A + \mathbb{Z}(A \cap \sigma)]$.

The following is an example of $\mathbb{N}A$ that does not satisfy the S_2 condition.
Example 1 (non-S_2).
$A = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4) = \begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$. Then

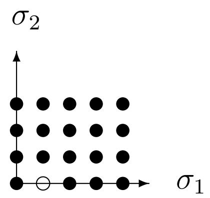

Figure 1. The semigroup $\mathbb{N}A$

In this case,

$$\mathbb{N}A = \mathbb{N}^2 \setminus \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{whereas} \quad \bigcap_{\sigma\colon \text{facet of } \mathbb{R}_{\geq 0}A} [\mathbb{N}A + \mathbb{Z}(A \cap \sigma)] = \mathbb{N}^2.$$

Theorem 3.4 ([19]).

$$\operatorname{Gr} D(R_A) \textit{ is Noetherian} \quad \Leftrightarrow \quad \mathbb{N}A \textit{ is } \textbf{scored}.$$

Let $\mathcal{F}$ be the set of facets of $\mathbb{R}_{\geq 0}A$. For a facet $\sigma \in \mathcal{F}$, we define the **primitive integral support function** F_σ of σ as the linear form on $\mathbb{R}^d$ uniquely determined by the conditions:
(1) $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$,
(2) $F_\sigma(\sigma) = 0$,
(3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

Definition 3.5. The semigroup $\mathbb{N}A$ is said to be **scored** if

$$\mathbb{N}A = \bigcap_{\sigma:\text{facet}} \{\, \boldsymbol{a} \in \mathbb{Z}^d \;:\; F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

Remark 3.6.

$$\mathbb{N}A\text{: scored} \quad \Rightarrow \quad \mathbb{N}A\text{: S}_2.$$

Proof. For each facet σ,

$$\mathbb{N}A \subseteq \mathbb{N}A + \mathbb{Z}(A \cap \sigma) \subseteq \{\, \boldsymbol{a} \in \mathbb{Z}^d \,:\, F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

Hence

$$\mathbb{N}A \subseteq \bigcap_{\sigma \in \mathcal{F}} (\mathbb{N}A + \mathbb{Z}(A \cap \sigma)) \subseteq \bigcap_{\sigma \in \mathcal{F}} \{\, \boldsymbol{a} \in \mathbb{Z}^d \,:\, F_\sigma(\boldsymbol{a}) \in F_\sigma(\mathbb{N}A) \,\}.$$

□

Example 2 (Scored).

$A_3 = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}$. Then

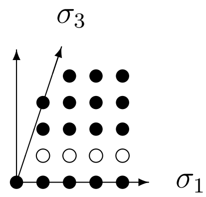

FIGURE 2. The semigroup $\mathbb{N}A_3$

$\mathcal{F} = \{\, \sigma_1 = \mathbb{R}_{\geq 0}\boldsymbol{a}_1,\ \sigma_3 = \mathbb{R}_{\geq 0}\boldsymbol{a}_3 \,\}$,
$F_{\sigma_1}(s_1, s_2) = s_2$, $F_{\sigma_3}(s_1, s_2) = 3s_1 - s_2$.
$F_{\sigma_1}(\mathbb{N}A) = \mathbb{N} \setminus \{1\}$, $F_{\sigma_3}(\mathbb{N}A) = \mathbb{N}$.
We have

$$\mathbb{N}A = \{\boldsymbol{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\boldsymbol{a}) \in \mathbb{N} \setminus \{1\},\ F_{\sigma_3}(\boldsymbol{a}) \in \mathbb{N}\}.$$

Hence $\mathbb{N}A$ is scored.

Example 3 (S_2 but non-scored). $A_2 = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Then

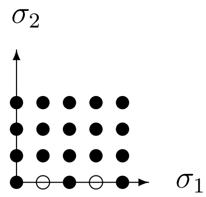

FIGURE 3. The semigroup $\mathbb{N}A_2$

$\mathcal{F} = \{\, \sigma_1 = \mathbb{R}_{\geq 0}\boldsymbol{a}_1,\ \sigma_2 = \mathbb{R}_{\geq 0}\boldsymbol{a}_2 \,\}$,
$F_{\sigma_1}(s_1, s_2) = s_2$, $F_{\sigma_2}(s_1, s_2) = s_1$.
$F_{\sigma_1}(\mathbb{N}A) = \mathbb{N}$, $F_{\sigma_3}(\mathbb{N}A) = \mathbb{N}$.
We have

$$\mathbb{N}A \subsetneq \{\boldsymbol{a} \in \mathbb{Z}^2 \mid F_{\sigma_1}(\boldsymbol{a}) \in \mathbb{N},\ F_{\sigma_3}(\boldsymbol{a}) \in \mathbb{N}\} = \mathbb{N}^2.$$

Hence $\mathbb{N}A$ is not scored.

The Running Example-1.
$d = 1$, $n = 2$, $A = (2, 3)$.
This is the smallest non-trivial example; we use this as a running example.
We have the following:

- $\mathbb{N}A = \{0, 2, 3, 4, \ldots\} = \mathbb{N} \setminus \{1\}$. $\mathbb{R}_{\geq 0}A = \mathbb{R}_{\geq 0}$.
- $\mathcal{F} = \{\{0\}\}$, $F_{\{0\}}(s) = s$; $\mathbb{N}A$ is scored.
- $R_A = \mathbb{C}[t^2, t^3]$.
- $D(R_A) = \{P \in \mathbb{C}[t^{\pm 1}]\langle \partial \rangle \,:\, P(\mathbb{C}[t^2, t^3]) \subseteq \mathbb{C}[t^2, t^3]\}$.
- $D(R_A) = \bigoplus_{a \in \mathbb{Z}} D(R_A)_a$, where
$$D(R_A)_a = \{P = \sum_{k \in \mathbb{Z}, l \in \mathbb{N},\, k-l=a} c_{k,l} t^k \partial^l \in D(R_A)\}.$$

4. The spectrum

By Theorem 3.4, the spectrum of $\operatorname{Gr} D(R_A)$ is in question, when $\mathbb{N}A$ is scored.

4.1. **Weight Decomposition.** It is easy to see $s_i := t_i \partial_i \in D(R_A)$ $(i = 1, \ldots, d)$.
For $\boldsymbol{a} = {}^t(a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$, set
$$D(R_A)_{\boldsymbol{a}} := \{P \in D(R_A) \,:\, [s_i, P] = a_i P \quad \text{for } i = 1, 2, \ldots, d\}.$$
Then $t_i \in D(R_A)_{\boldsymbol{e}_i}$, $\quad \boldsymbol{e}_i = {}^t(0, \ldots, \overset{i}{\check{1}}, \ldots, 0)$.

Lemma 4.1. (1) $D(R_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(R_A)_{\boldsymbol{a}}$.
(2) $D_k(R_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D_k(R_A) \cap D(R_A)_{\boldsymbol{a}}$.
(3) $\operatorname{Gr} D(R_A) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \operatorname{Gr} D(R_A)_{\boldsymbol{a}}$.

Theorem 4.2 ([11])**.**
$$D(R_A)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega(\boldsymbol{a})) \qquad \textit{for all } \boldsymbol{a} \in \mathbb{Z}^d,$$
where
$$\begin{aligned} \Omega(\boldsymbol{a}) &:= \{\, \boldsymbol{b} \in \mathbb{N}A : \ \boldsymbol{b} + \boldsymbol{a} \notin \mathbb{N}A \,\} = \mathbb{N}A \setminus (-\boldsymbol{a} + \mathbb{N}A), \\ \mathbb{I}(\Omega(\boldsymbol{a})) &:= \{f(s) \in \mathbb{C}[s] := \mathbb{C}[s_1, \ldots, s_d] \,:\, f \textit{ vanishes on } \Omega(\boldsymbol{a})\}. \end{aligned}$$
In particular, $D(R_A)_{\boldsymbol{0}} = \mathbb{C}[s]$.

The Running Example-2.
$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$.
$a \in \mathbb{Z}$. $\Omega(a) = \mathbb{N}A \setminus (-a + \mathbb{N}A)$. $\qquad D(R_A)_a = t^a \mathbb{I}(\Omega(a))$.

- $\Omega(a) = \emptyset$ $(a \in \mathbb{N}A)$, $\qquad D(R_A)_a = t^a \mathbb{C}[s]$.
- $\Omega(1) = \{0\}$, $\qquad D(R_A)_1 = ts\mathbb{C}[s] = t^2 \partial \mathbb{C}[s]$.
- $\Omega(-1) = \{0, 2\}$, $\qquad D(R_A)_{-1} = t^{-1} s(s-2)\mathbb{C}[s]$.
- $\Omega(-2) = \{0, 3\}$, $\qquad D(R_A)_{-2} = t^{-2} s(s-3)\mathbb{C}[s]$.
- $\Omega(-k) = \{0, 2, \ldots, k-1\} \cup \{k+1\}$ $(k \geq 3)$,
$$D(R_A)_{-k} = t^{-k} s(s-2) \cdots (s-(k-1))(s-(k+1))\mathbb{C}[s].$$

Note that $|\Omega(-k)| = k$ if $k \in \mathbb{N}A$.

4.2. **$\mathbb{Z}^d$-graded Prime Ideals.** From now on, we assume that $\mathbb{N}A$ **is scored**, and set $G := \operatorname{Gr} D(R_A)$. By Lemma 4.1, we work on $\mathbb{Z}^d$-graded prime ideals of G.

Corollary 4.3 (to Theorem 4.2)**.**

$$G = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} \overline{t^{\boldsymbol{a}}\mathbb{I}(\Omega(\boldsymbol{a}))} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d} \overline{P}_{\boldsymbol{a}}\mathbb{C}[s],$$

where

$$\begin{aligned} p_{\boldsymbol{a}} &:= \prod_{\sigma} \prod_{m\in F_\sigma(\mathbb{N}A)\setminus(-F_\sigma(\boldsymbol{a})+F_\sigma(\mathbb{N}A))} (F_\sigma(s)-m), \\ P_{\boldsymbol{a}} &:= t^{\boldsymbol{a}}\cdot p_{\boldsymbol{a}}(s), \\ \overline{P}_{\boldsymbol{a}} &= t^{\boldsymbol{a}}\cdot \prod_{\sigma} F_\sigma(s)^{\sharp(F_\sigma(\mathbb{N}A)\setminus(-F_\sigma(\boldsymbol{a})+F_\sigma(\mathbb{N}A)))}. \end{aligned}$$

Since $G_{\mathbf{0}} = \mathbb{C}[s]$ is a subalgebra of G, the following lemma is immediate.

Lemma 4.4. *Let $\mathfrak{P} = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^d}\mathfrak{P}_{\boldsymbol{a}}$ be a $\mathbb{Z}^d$-graded prime ideal of G. Then $\mathfrak{P}_{\mathbf{0}}$ is a prime ideal of $G_{\mathbf{0}} = \mathbb{C}[s]$.*

Given a prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we shall classify all $\mathbb{Z}^d$-graded prime ideals $\mathfrak{P}$ of G with $\mathfrak{P}_{\mathbf{0}} = \mathfrak{p}$.

4.3. **Degree and Expected Degree.** For $\sigma\in\mathcal{F}$ and $\boldsymbol{a}\in\mathbb{Z}^d$, set

- $\deg_\sigma(\boldsymbol{a}) := \sharp\,(F_\sigma(\mathbb{N}A)\setminus(-F_\sigma(\boldsymbol{a})+F_\sigma(\mathbb{N}A)))$,
- $\operatorname{expdeg}_\sigma(\boldsymbol{a}) := \begin{cases} 0 & \text{if } F_\sigma(\boldsymbol{a})\ge 0 \\ |F_\sigma(\boldsymbol{a})| & \text{if } F_\sigma(\boldsymbol{a})\le 0. \end{cases}$

Then

$$\overline{P_{\boldsymbol{a}}} = t^{\boldsymbol{a}}\prod_{\sigma\in\mathcal{F}} F_\sigma^{\deg_\sigma(\boldsymbol{a})}.$$

The Running Example-3.

$A = (2,3)$, $\quad \mathbb{N}A = \mathbb{N}\setminus\{1\}$.

$F_{\{0\}}(s) = s$.

a	$\cdots$	$-k$	$\cdots$	-3	-2	-1	0	1	2	3	$\cdots$
$\operatorname{expdeg}_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	1	0	0	0	0	$\cdots$
$\deg_{\{0\}}(a)$	$\cdots$	k	$\cdots$	3	2	**2**	0	**1**	0	0	$\cdots$

$$G = \bigoplus_{a\in\mathbb{Z}} t^a s^{\deg_{\{0\}}(a)}\mathbb{C}[s] \subseteq \mathbb{C}[t^{\pm1},\xi], \qquad s = t\xi.$$

For a fixed prime ideal $\mathfrak{p}$ of $\mathbb{C}[s]$, we define

$$\begin{aligned} \mathcal{F}(\mathfrak{p}) &:= \{\sigma\in\mathcal{F} : F_\sigma\in\mathfrak{p}\}, \\ \Sigma(\mathfrak{p}) &: \text{the fan determined by the hyperplane arrangement } \{\mathbb{R}\sigma : \sigma\in\mathcal{F}(\mathfrak{p})\}, \\ S(\mathfrak{p}) &:= \{\boldsymbol{a}\in\mathbb{Z}^d : |F_\sigma(\boldsymbol{a})|\in F_\sigma(\mathbb{N}A) \quad (\text{for } \forall\sigma\in\mathcal{F}(\mathfrak{p}))\}. \end{aligned}$$

The Running Example-4.
$A = (2,3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}$.

$F_{\{0\}}(s) = s$.

- $\mathfrak{p} = (s - \beta)$: a fixed prime ideal of $\mathbb{C}[s]$
- $\mathcal{F}((s-\beta)) = \{\,\sigma \in \mathcal{F} \,:\, F_\sigma \in (s-\beta)\} = \begin{cases} \{0\} & (\beta = 0) \\ \emptyset & (\text{otherwise}). \end{cases}$
- $\Sigma((s-\beta)) = \begin{cases} \{\mathbb{R}_{\geq 0}, \{0\}, \mathbb{R}_{\leq 0}\} & (\beta = 0) \\ \{\mathbb{R}\} & (\text{otherwise}). \end{cases}$
- $S((s-\beta)) = \begin{cases} \mathbb{Z} \setminus \{\pm 1\} & (\beta = 0) \\ \mathbb{Z} & (\text{otherwise}). \end{cases}$

For $\boldsymbol{a} \in \mathbb{Z}^d$, put

- $\deg_{\mathfrak{p}}(\boldsymbol{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \deg_\sigma(\boldsymbol{a})$.
- $\operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a}) := \sum_{\sigma \in \mathcal{F}(\mathfrak{p})} \operatorname{expdeg}_\sigma(\boldsymbol{a})$.

Then $\deg_{\boldsymbol{m}}(\boldsymbol{a}) = \deg(p_{\boldsymbol{a}})$, where $\boldsymbol{m} = (s_1, \ldots, s_d)$.

Proposition 4.5. (1) $\deg_{\mathfrak{p}}(\boldsymbol{a}) \geq \operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a})$.
(2) $\deg_{\mathfrak{p}}(\boldsymbol{a}) = \operatorname{expdeg}_{\mathfrak{p}}(\boldsymbol{a})$ *if and only if* $\boldsymbol{a} \in S(\mathfrak{p})$.

4.4. **Classification.** For a cone $\tau \in \Sigma(\mathfrak{p})$, we define an ideal $\mathfrak{P}(\mathfrak{p}, \tau) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} \mathfrak{P}(\mathfrak{p}, \tau)_{\boldsymbol{a}}$ of G by

$$\mathfrak{P}(\mathfrak{p}, \tau)_{\boldsymbol{a}} := \begin{cases} G_{\boldsymbol{a}}\mathfrak{p} & (\boldsymbol{a} \in \tau \cap S(\mathfrak{p})) \\ G_{\boldsymbol{a}} & (\text{otherwise}). \end{cases}$$

Proposition 4.6. *The* $\mathbb{Z}^d$*-graded ideal* $\mathfrak{P}(\mathfrak{p}, \tau)$ *is prime.*

Theorem 4.7 ([17])**.** *Let* $\mathfrak{P}$ *be a* $\mathbb{Z}^d$*-graded prime ideal with* $\mathfrak{P}_{\mathbf{0}} = \mathfrak{p}$*. Then there exists* $\tau \in \Sigma(\mathfrak{p})$ *such that* $\mathfrak{P} = \mathfrak{P}(\mathfrak{p}, \tau)$.

Proposition 4.8. $\mathfrak{P}(\mathfrak{p}, \tau) \subseteq \mathfrak{P}(\mathfrak{p}', \tau')$ *if and only if* $\mathfrak{p} \subseteq \mathfrak{p}'$ *and* $\tau \supseteq \tau'$.

Proposition 4.9. $\dim G/\mathfrak{P}(\mathfrak{p}, \tau) = \dim \mathbb{C}[s]/\mathfrak{p} + \dim \tau$.

The Running Example-5.
$A = (2,3), \quad \mathbb{N}A = \mathbb{N} \setminus \{1\}. \qquad$ Let $a \in \mathbb{Z}$.

- $\mathfrak{P}((s), \mathbb{R}_{\geq 0})_a = \begin{cases} G_a s & (a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$
- $\mathfrak{P}((s), \{0\})_a = \begin{cases} G_a s & (a = 0) \\ G_a & (a \neq 0). \end{cases}$
- $\mathfrak{P}((s), \mathbb{R}_{\leq 0})_a = \begin{cases} G_a s & (-a \in \mathbb{N} \setminus \{1\}) \\ G_a & (\text{otherwise}). \end{cases}$

$$\mathfrak{P}((s), \mathbb{R}_{\geq 0}) \subseteq \mathfrak{P}((s), \{0\}) \supseteq \mathfrak{P}((s), \mathbb{R}_{\leq 0}).$$

- $\mathfrak{P}((s-\beta), \mathbb{R})_a = G_a(s-\beta) \quad (\forall a \in \mathbb{Z}) \quad$ for $\beta \neq 0$.

5. Characteristic Variety

5.1. **Critical Modules.** We denote by $\operatorname{Kdim} M$ the **Krull dimension** for the lattice of $\mathbb{Z}^d$-graded $D(R_A)$-submodules in the sense of Rentschler and Gabriel ([2], [13]).

The Krull dimension is defined inductively:

- $\operatorname{Kdim} M = 0 \overset{\text{def.}}{\Leftrightarrow} M$: Artinian
- $\operatorname{Kdim} M = \delta$ if
  - $\operatorname{Kdim} M \neq \delta'$ for $\delta' < \delta$.
  - for every descending chain of $\mathbb{Z}^d$-graded $D(R_A)$-submodules
$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$
    $\operatorname{Kdim} M_i/M_{i+1} < \delta$ for all but finitely many i.

Definition 5.1. For a $\mathbb{Z}^d$-graded $D(R_A)$-module M of Krull dimension δ,

$$M \text{ is } \delta\textbf{-critical} \overset{\text{def.}}{\Leftrightarrow} \begin{array}{r} \operatorname{Kdim}(M/N) < \delta \text{ for all nonzero } \mathbb{Z}^d\text{-graded} \\ D(R_A)\text{-submodules } N \text{ of } M. \end{array}$$

Then the 0-critical modules are exactly the simple modules. In this sense, a critical module is a generalization of a simple module. The notions of Krull dimension and critical modules enable us to use Artinian type method to Noetherian rings (see e.g. [5], [9], [10]).

Remark 5.2. Let R be a commutative Noetherian ring, and M a finitely generated R-module. Then

M is critical $\Leftrightarrow \exists \mathfrak{p} \in \operatorname{Spec}(R)$ s.t. $\operatorname{Ass}(M) = \{\mathfrak{p}\}$, and $\operatorname{length}_{R_\mathfrak{p}}(M_\mathfrak{p}) = 1$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{C}[s]$, and let $\delta = \dim \mathbb{C}[s]/\mathfrak{p}$. Define a $\mathbb{Z}^d$-graded $D(R_A)$-module $L(\mathfrak{p}) := D(R_A)/I(\mathfrak{p})$ by

$$I(\mathfrak{p})_{\boldsymbol{a}} := \begin{cases} D(R_A)_{\boldsymbol{a}}\mathfrak{p} & (\mathfrak{p} \sim \mathfrak{p} + \boldsymbol{a}) \\ D(R_A)_{\boldsymbol{a}} & (\text{otherwise}), \end{cases}$$

where

- $\mathfrak{p} \sim \mathfrak{p} + \boldsymbol{a} \overset{\text{def.}}{\Leftrightarrow} \mathbb{I}(\Omega(\boldsymbol{a})) \not\subseteq \mathfrak{p}$ and $\mathbb{I}(\Omega(-\boldsymbol{a})) \not\subseteq \mathfrak{p} + \boldsymbol{a}$,
- $\mathfrak{p} + \boldsymbol{a} = \{f(s - \boldsymbol{a}) \,:\, f(s) \in \mathfrak{p}\}$.

Then $L(\mathfrak{p})$ is δ-critical [16].

The Running Example-6.

$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$. Let $\beta \in \mathbb{C}$, $a \in \mathbb{Z}$.

- $\mathfrak{p} = (s - \beta)$, $\delta = 0$.
- Let $\beta \notin \mathbb{Z}$. Then $(s - \beta) \sim (s - \beta) + a$ for $\forall a \in \mathbb{Z}$.
  $I((s - \beta))_a = D(R_A)_a(s - \beta)$ for $\forall a \in \mathbb{Z}$.
- Let $\beta \in \mathbb{N}A$. Then $(s - \beta) \sim (s - \beta) + a \Leftrightarrow \beta + a \in \mathbb{N}A$.
$$I((s-\beta))_a = \begin{cases} D(R_A)_a(s - \beta) & (\beta + a \in \mathbb{N}A) \\ D(R_A)_a & (\beta + a \notin \mathbb{N}A). \end{cases}$$
- Let $\beta \in \mathbb{Z} \setminus \mathbb{N}A$. Then $(s - \beta) \sim (s - \beta) + a \Leftrightarrow \beta + a \notin \mathbb{N}A$.
$$I((s-\beta))_a = \begin{cases} D(R_A)_a(s - \beta) & (\beta + a \notin \mathbb{N}A) \\ D(R_A)_a & (\beta + a \in \mathbb{N}A). \end{cases}$$

Theorem 5.3 ([16])**.** *Let M be a δ-critical $\mathbb{Z}^d$-graded left $D(R_A)$-module singly generated by $v \in M_{\boldsymbol{0}}$ with $\operatorname{Ann}_{\mathbb{C}[s]}(v) = \mathfrak{p}$. Then M is isomorphic to $L(\mathfrak{p})$.*

5.2. **Characteristic Varieties.** For a cyclic $D(R_A)$-module $D(R_A)/I$, the support of the G-module $G/\mathrm{Gr}I$, where $\mathrm{Gr}I = \bigoplus_{k=0}^{\infty} I \cap D_k(R_A)/I \cap D_{k-1}(R_A)$, is called the *characteristic variety* $\mathrm{Ch}(D(R_A)/I)$ of $D(R_A)/I$. For details about characteristic varieties, see any textbook of the theory of D-modules, for example, [7], [8], etc.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{C}[s]$ **homogeneous with respect to** $s_1, \dots, s_d$.

Define $\tau(\mathfrak{p} + \boldsymbol{\beta})$ by

$$\tau(\mathfrak{p} + \boldsymbol{\beta}) := \bigcap_{\sigma \in \mathcal{F}(\mathfrak{p});\, F_\sigma(\boldsymbol{\beta}) \in F_\sigma(\mathbb{N}A)} (F_\sigma \geq 0) \cap \bigcap_{\sigma \in \mathcal{F}(\mathfrak{p});\, F_\sigma(\boldsymbol{\beta}) \in \mathbb{Z} \setminus F_\sigma(\mathbb{N}A)} (F_\sigma \leq 0).$$

Then $\tau(\mathfrak{p} + \boldsymbol{\beta})$ is a union of cones in $\Sigma(\mathfrak{p})$.

Theorem 5.4 ([17]). (1) $\sqrt{\mathrm{Gr}\, I(\mathfrak{p} + \boldsymbol{\beta})} = \bigcap_{\tau \in \Sigma(\mathfrak{p}),\, \tau \subseteq \tau(\mathfrak{p}+\boldsymbol{\beta})} \mathfrak{P}(\mathfrak{p}, \tau)$. *Hence*

$$\mathrm{Ch}(L(\mathfrak{p} + \boldsymbol{\beta})) = \bigcup_{\tau \in \Sigma(\mathfrak{p}),\, \tau \subseteq \tau(\mathfrak{p}+\boldsymbol{\beta})} \mathrm{Supp}(G/\mathfrak{P}(\mathfrak{p}, \tau)).$$

(2) *The characteristic variety of $L(\mathfrak{p} + \boldsymbol{\beta})$ is irreducible if and only if $\tau(\mathfrak{p} + \boldsymbol{\beta}) \in \Sigma(\mathfrak{p})$. In this case,*

$$\sqrt{\mathrm{Gr}\, I(\mathfrak{p} + \boldsymbol{\beta})} = \mathfrak{P}(\mathfrak{p}, \tau(\mathfrak{p} + \boldsymbol{\beta})).$$

(3) *If $F_\sigma(\boldsymbol{\beta}) \in \mathbb{Z}$ for all $\sigma \in \mathcal{F}(\mathfrak{p})$, then the characteristic variety of $L(\mathfrak{p} + \boldsymbol{\beta})$ is irreducible.*

(4) $\mathrm{Gr}\, I(\mathfrak{p}) = \mathfrak{P}(\mathfrak{p}, \tau(\mathfrak{p}))$.

The Running Example-7.

$A = (2, 3)$, $\mathbb{N}A = \mathbb{N} \setminus \{1\}$. $F_{\{0\}}(s) = s$. $(s) + \beta = (s - \beta)$.

$$\tau((s - \beta)) = \begin{cases} \mathbb{R}_{\geq 0} & (\beta \in \mathbb{N}A) \\ \mathbb{R}_{\leq 0} & (\beta \in \mathbb{Z} \setminus \mathbb{N}A) \\ \mathbb{R} & (\beta \notin \mathbb{Z}). \end{cases}$$

- $\mathrm{Gr}\, I((s - \beta)) = \mathfrak{P}((s), \mathbb{R}_{\geq 0}) \qquad (\beta \in \mathbb{N}A)$.
- $\sqrt{\mathrm{Gr}\, I((s - \beta))} = \mathfrak{P}((s), \mathbb{R}_{\leq 0}) \qquad (\beta \in \mathbb{Z} \setminus \mathbb{N}A)$.
- $\sqrt{\mathrm{Gr}\, I((s - \beta))} = \mathfrak{P}((s), \mathbb{R}_{\geq 0}) \cap \mathfrak{P}((s), \mathbb{R}_{\leq 0}) \qquad (\beta \notin \mathbb{Z})$.

Example.

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$. There are four facets: σ_{14}, σ_{36}, σ_{123}, σ_{456}. The primitive integral support functions are $F_{14}(s) = s_2$, $F_{36}(s) = 3s_1 - s_2$, $F_{123}(s) = s_3$, $F_{456}(s) = s_1 - s_3$.

$$F_{14}(\mathbb{N}A) = \mathbb{N} \setminus \{1\}, \qquad F_{36}(\mathbb{N}A) = \mathbb{N},$$
$$F_{123}(\mathbb{N}A) = \mathbb{N}, \qquad F_{456}(\mathbb{N}A) = \mathbb{N}.$$

$$\begin{aligned} \tau(\mathfrak{m} + {}^t[0,1,0]) &= (F_{14} \leq 0) \cap (F_{36} \geq 0) \cap (F_{123} \geq 0) \cap (F_{456} \geq 0) \\ &= \{\mathbf{0}\}, \qquad \text{where } \mathfrak{m} = (s_1, s_2, s_3). \end{aligned}$$

$$\sqrt{\mathrm{Gr}\, I(\mathfrak{m} + {}^t[0,1,0])} = \mathfrak{P}(\mathfrak{m}, \{\mathbf{0}\}).$$

Hence $\dim \mathrm{Ch}(L(\mathfrak{m} + {}^t[0,1,0])) = 0$.

Theorem 5.5. *Suppose that $\mathbb{N}A$ is scored. Then the following are equivalent:*

(1) $\dim \mathrm{Ch}(M) \geq d$ *for all nonzero finitely generated $\mathbb{Z}^d$-graded $D(R_A)$-modules M.*
(2) *$D(R_A)$ is simple.*
(3) *For all $\boldsymbol{\beta} \in \mathbb{C}^d$,*

$$\left\{ \gamma \in \mathbb{R}^d \mid \begin{array}{ll} F_\sigma(\gamma) > 0 & (\forall \sigma \in \mathcal{F}_+(\boldsymbol{\beta})) \\ F_\sigma(\gamma) < 0 & (\forall \sigma \in \mathcal{F}_-(\boldsymbol{\beta})) \end{array} \right\} \neq \emptyset, \tag{5.1}$$

where

$$\begin{aligned} \mathcal{F}_+(\boldsymbol{\beta}) &= \{\sigma \in \mathcal{F} \mid F_\sigma(\boldsymbol{\beta}) \in F_\sigma(\mathbb{N}A)\}, \\ \mathcal{F}_-(\boldsymbol{\beta}) &= \{\sigma \in \mathcal{F} \mid F_\sigma(\boldsymbol{\beta}) \in \mathbb{Z} \setminus F_\sigma(\mathbb{N}A)\}. \end{aligned}$$

Proof. (2) $\Leftrightarrow$ (3). This is [15, Theorem 7.25].

(1) $\Leftrightarrow$ (3). The condition (1) is equivalent to the condition:

$$\dim \mathrm{Ch}(L) \geq d \text{ for all simple } \mathbb{Z}^d\text{-graded } D(R_A)\text{-modules } L. \tag{5.2}$$

Any simple $\mathbb{Z}^d$-graded $D(R_A)$-module is of the form $L(\mathfrak{m}+\boldsymbol{\beta})$ [12, Proposition 3.17], where $\mathfrak{m} = (s_1, s_2, \ldots, s_d)$ and $\boldsymbol{\beta} \in \mathbb{C}^d$. By Proposition 4.9 and Theorem 5.4 (1), $\dim \mathrm{Ch}(L(\mathfrak{m} + \boldsymbol{\beta})) = \dim \tau(\mathfrak{m} + \boldsymbol{\beta})$. Clearly $\dim \tau(\mathfrak{m} + \boldsymbol{\beta}) = d$ if and only if (5.1) is satisfied. □

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